Numerical methods for fully nonlinear free surface water waves

Denys Dutykh, Claudio Viotti

Numerical methods for fully nonlinear free surface waves

Denys Dutykh[1] & Claudio Viotti[1]

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Short Course on "*Modeling of Nonlinear Ocean Waves*"

Outline of the short course

1. **Lecture 1**
   - Introduction (D.D.)
   - BIEM (C.V.)
   - Spectral CG-method (D.D.)

2. **Lecture 2**
   - Higher-Order Spectral (HOS) methods (D.D.)
   - Dirichlet-to-Neumann (D2N) operator technique (D.D.)
   - Conformal mappings (C.V.)

Remark:

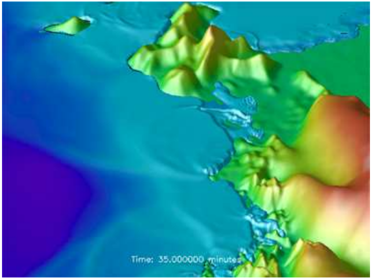
Focus only on numerical methods for the full Euler equations

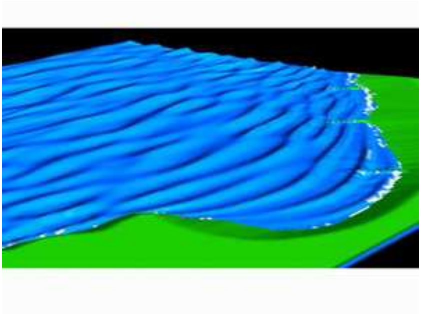
- <u>No</u> asymptotic models

What we will not cover in this course:

Self-learning is your friend!

- The first modern spectral method for water waves by Rienecker & Fenton (1981-1982) [RF81, FR82]
- Methods based on eigenfunctions expansions
  - Coupled-mode approach by Belibassakis & Athanassoulis [AB99, BA06]
- Finite difference-based methods
- Any kind of meshless methods





PHYSICAL ASSUMPTIONS:

- Fluid is ideal (inviscid) ($Re = \infty$)
- Fluid is homogeneous ($\rho_w = const$)
- Flow is incompressible ($\nabla \cdot \underline{u} = 0$)
- Flow is potential ($\underline{u} = \nabla\phi$)

INTERFACE CONDITIONS:

- Interface is a graph: $y = \eta(\underline{x}, t)$
- Air effect is neglected ($\rho_a \ll \rho_w$)
- Free surface is isobaric

BASIC MODEL:

- Incompressible Euler equations with free surface

Euler equations

with free surface

- Incompressibility:

$$\nabla \cdot \underline{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

- Momentum conservation:

$$\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} + \frac{\nabla p}{\rho} = \underline{g}$$

$$\phi_t + \frac{1}{2}|\nabla \phi|^2 + gz + \frac{p}{\rho} = B(t)$$

Boundary conditions:

$$\eta_t + \underline{u} \cdot \nabla \eta = v, \quad z = \eta(\underline{x}, t)$$

$$p = 0, \quad z = \eta(\underline{x}, t)$$

$$v_n = \underline{u} \cdot \underline{n}_b = 0, \quad z = -d(\underline{x})$$

Water wave problem

Potential flow formulation

- Continuity equation

$$\nabla^2\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d(\underline{x}), \eta(\underline{x}, t)],$$

- Kinematic bottom condition

$$\frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial y}, \quad y = \eta(\underline{x}, t),$$

- Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$

FIGURE: Laplace

HAMILTONIAN STRUCTURE

PETROV (1964) [PET64]; ZAKHAROV (1968) [ZAK68]; CRAIG & SULEM (1993) [CS93]

CANONICAL VARIABLES:

$\eta(\underline{x}, t)$: free surface elevation

$\varphi(\underline{x}, t)$: velocity potential at the free surface

$\varphi(\underline{x}, t) := \phi(\underline{x}, y = \eta(\underline{x}, t), t)$

- Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \varphi}, \qquad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

- Hamiltonian:

$$\mathscr{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla \phi|^2 \,\mathrm{d}y + \frac{1}{2} g \eta^2$$

APPLICATION TO NUMERICS:

Used in D2N operator methods: [CS93, GN07, XG09]

LUKE'S VARIATIONAL PRINCIPLES

J.C. LUKE, JFM (1967) [LUK67]

- First improvement of the classical Lagrangian $\mathscr{L} := K - \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, \mathrm{d}\underline{x} \, \mathrm{d}t, \qquad \mathscr{L} := \int_{-d}^{\eta} \Big(\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy \Big) \, \mathrm{d}y$$

$$\begin{aligned}
\delta\phi &: \; \Delta\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d, \eta], \\
\delta\phi|_{y=-d} &: \; \frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d, \\
\delta\phi|_{y=\eta} &: \; \frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta - \frac{\partial \phi}{\partial y} = 0, \quad y = \eta(\underline{x}, t), \\
\delta\eta &: \; \frac{\partial \phi}{\partial t} + \tfrac{1}{2} |\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).
\end{aligned}$$

- Water wave problem formulation is recovered from $\mathcal{L}$

APPLICATION TO NUMERICS:

- Not fully explored...
- Coupled-mode technique by B. & A. (2006) [BA06]

Why this problem is difficult?

Outline of some numerical difficulties

- Problem is highly nonlinear
- Computational domain is unknown ($y = \eta(\underline{x}, t)$ to be determined)
- Formulation is stiff (Hamiltonian structure)
- Taylor expansions involve very high derivatives
- Physical and numerical instabilities
- No dissipation to stabilize computation
- Overturning surface…

To be continued by Claudio...

http://www.denys-dutykh.com/

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J. Fluid Mech., 104:119–137, April 1981.

L. Xu and P. Guyenne.
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J. Appl. Mech. Tech. Phys., 9:190–194, 1968.

Boundary Element Method for Three Dimensional Water Waves

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Credits

- Grilli, Skourup & Svendsen 1989 [1]
- Grilli, Guyenne & Dias 2001 [2]
- Fochesato & Dias 2006 [3]

Range of applicability

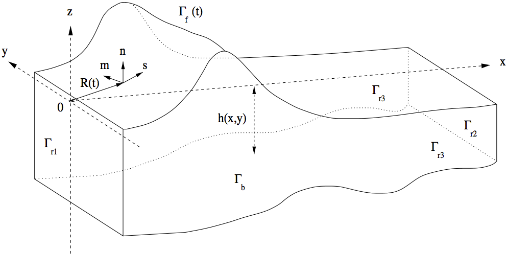
This numerical approach is probably the most general among those based on potential flow theory...

- 2D & 3D
- Overturning waves (up to wave breaking)
- Arbitrary bathymetry
- Arbitray boundary conditions (wave makers, absorbing beach, ...)

...of course, this comes at a cost

- Complicated
- Expensive

Governing equations and boundary conditions



On the free surface:

$$\frac{D\mathbf{R}}{Dt} = \mathbf{u} = \nabla\phi$$
$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla\phi|^2 - g\mathbf{z} - \frac{p_a}{\rho}$$

where:

$$\frac{D\phi}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

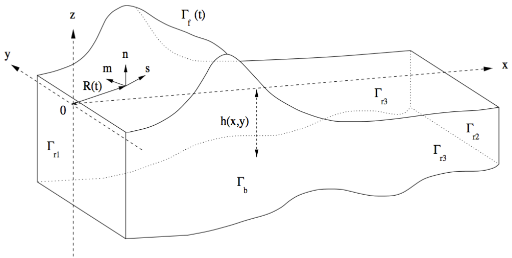
On the other boundaries:

$$\nabla\phi\cdot\mathbf{n} = 0 \quad \text{or} \quad \nabla\phi\cdot\mathbf{n} = V_b$$

Inside the fluid volume Ω:

$$\nabla^2\phi = 0$$

For time marching need to find $\nabla\phi$ from the knowledge of ϕ.



Laplace equation for the potential

$$\nabla^2 \phi(\mathbf{x}) = 0,$$

with boundary conditions

$$\begin{cases} \phi = \overline{\phi} & \text{on} \quad \Gamma_f(t) \quad \text{(free surface)} \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on} \quad \Gamma_b \quad \text{(bottom)} \\ \dfrac{\partial \phi}{\partial n} = V_i & \text{on} \quad \Gamma_i, \quad i = 1, 2, 3 \quad \text{(lateral boundaries).} \end{cases}$$

Green's function

Free space Green's function

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}'), \qquad \lim_{|\mathbf{x}|\to\infty} G = 0$$

3D Green's function

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi r}, \quad \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}, \quad \mathbf{r} = \mathbf{x}-\mathbf{x}', \quad r = |\mathbf{r}|.$$

Combine with $\nabla^2\phi = 0$ to obtain $\nabla^2\phi G - \phi\nabla^2 G = -\delta\phi$.
Integrate over Ω:

$$\int_\Omega \nabla^2\phi(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x}')\nabla^2 G(\mathbf{x},\mathbf{x}')\mathrm{d}\mathbf{x}' = -\int_\Omega \delta(\mathbf{x}')\phi(\mathbf{x}')\mathrm{d}\mathbf{x}$$

$$\int_\Omega \nabla\cdot\left(\nabla\phi(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x}')\nabla G(\mathbf{x},\mathbf{x}')\right)\mathrm{d}\mathbf{x}' = -\int_\Omega \delta(\mathbf{x}')\phi(\mathbf{x}')\mathrm{d}\mathbf{x}$$

Then apply Divergence Theorem:

$$\int_\Gamma (\nabla\phi G - \phi\nabla G)\cdot\mathbf{n}\mathrm{d}\Gamma = -\phi$$

Boundary Integral Equation

$$\begin{aligned}
\alpha(\mathbf{x})\phi(\mathbf{x}) &= \int_\Gamma \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}')d\Gamma \\
&= \int_{\Gamma_s} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}')d\Gamma_s \\
&+ \int_{\Gamma_b} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}')d\Gamma_b \\
&+ \sum_i \int_{\Gamma_b} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{x}')d\Gamma_i
\end{aligned}$$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \text{ inside the domain} \\ 1/2 & \text{for } \mathbf{x} \text{ on a smooth part of the boundary} \\ 1/4 & \text{for } \mathbf{x} \text{ on a edge} \\ 1/8 & \text{for } \mathbf{x} \text{ on a corner} \end{cases}$$

BIE is solved for the unknown variables on the boundary: $\frac{\partial\phi}{\partial n}$ on the free surface, and ϕ on the other boundaries.

Boundary Element Method (BEM) Discetization

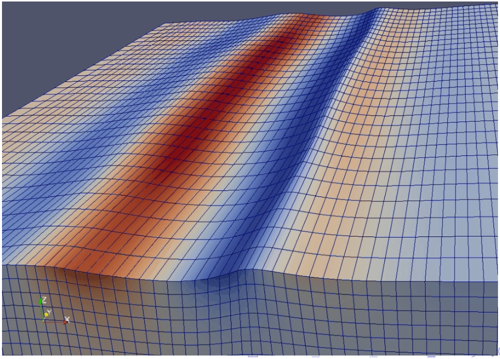
- N_Γ **collocation nodes** (i.e., points where all variables are sampled)
- M_Γ **finite elements** (discrete portions of the boundary, where all variables are reconstructed from neighboring nodal values)
- $k = 1, ..., M_\Gamma$: **boundary element** index
- $N_j(\xi, \eta)$: **shape function** associated with the j-th node.
- (ξ, η): local coordinates of the finite elements.

Represent geometry and field variable within each element:

$$\mathbf{x}(\xi, \eta) = N_j(\xi, \eta)\mathbf{x}_j^k,$$

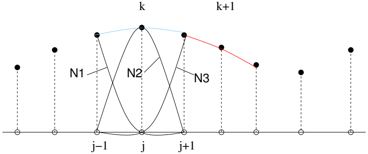
$$q(\xi, \eta) = N_j(\xi, \eta)q_j^k,$$

$j =$
$1, .., m$ (nodes of the k-th element)



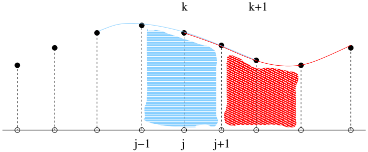
Construction of shape functions $N_j(\xi, \eta)$

The (too) simple way: polynomial interpolation using the nodal values which belong **only** to the current element



Inconvenient: only C^0 continuity is obtained, this was seen to enhance numerical instabilities (Grilli *et al.* 1989)

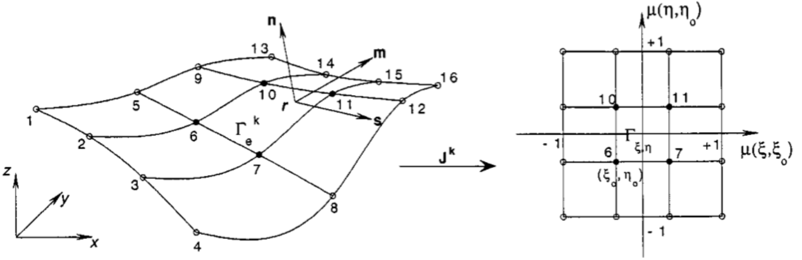
The better way: use also nodal values **outside** the current element: middle-interval-interpolation (MII)



Smoother reconstruction, enforce C^n regularity.

3D-MII Elements

Example (Grilli, Guyenne & Dias 2001): 16-node cubic 3D-MII element



$$N_j(\xi,\eta) = N'_{b(j)}(\mu(\xi,\xi_0))N'_{d(j)}(\mu(\eta,\eta_0)), \quad b,d = 1,..,4; \quad j = 4(d-1)+b$$

$$N'_1(\mu) = \tfrac{1}{16}(1-\mu)(9\mu^2-1), \quad N'_2(\mu) = \tfrac{9}{16}(1-\mu^2)(1-3\mu),$$

$$N'_3(\mu) = \tfrac{9}{16}(1-\mu^2)(1+3\mu), \quad N'_4(\mu) = \tfrac{1}{16}(1+\mu)(9\mu^2-1),$$

Discretization of the Boundary Integral Equation

BIE is discretized by using shape-functions reconstructions of the geometry and field variables, and summing the contributions from all finite elements:

$$\int_{\Gamma(x)} \frac{\partial \phi}{\partial n} G_l \, \mathrm{d}\Gamma = \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} \int_{\Gamma_{\xi,\eta}} N_j(\xi,\eta) G(\boldsymbol{x}(\xi,\eta), \boldsymbol{x}_l) \left| \mathbf{J}^k(\xi,\eta) \right| \mathrm{d}\xi \, \mathrm{d}\eta \right\} \frac{\partial \phi}{\partial n}(\boldsymbol{x}_j)$$
$$= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} D_{lj}^k \right\} \frac{\partial \phi_j}{\partial n} = \sum_{j=1}^{N_\Gamma} K_{lj}^d \frac{\partial \phi_j}{\partial n}$$

$$\int_{\Gamma(x)} \phi \frac{\partial G_l}{\partial n} \mathrm{d}\Gamma = \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} \int_{\Gamma_{\xi,\eta}} N_j(\xi,\eta) \frac{\partial G(\boldsymbol{x}(\xi,\eta), \boldsymbol{x}_l)}{\partial n} \left| \mathbf{J}^k(\xi,\eta) \right| \mathrm{d}\xi \, \mathrm{d}\eta \right\} \phi(\boldsymbol{x}_j)$$
$$= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} E_{lj}^k \right\} \phi_j = \sum_{j=1}^{N_\Gamma} K_{lj}^n \phi_j$$

(Remember: the Jacobian is obtained from $\mathbf{x}(\xi,\eta) = N_j(\xi,\eta)\mathbf{x}_j^k, \quad j = 1, ..., m$)

This procedure yields the Dirichlet and Neumann $N_\Gamma \times N_\Gamma$ matrices:

$$\mathbf{K}^d, \quad \mathbf{K}^n.$$

Some technical remarks

- *Singular integrals.* Integral's kernel (Green's function) is singular for $l = j$ in the previous formulae. Particular treatment must be applied in this case: singularity extraction.
- *Edges and corners.* Double-node scheme: nodes on the edges are "counted twice". The value of ϕ is imposed to be the same, whereas $\frac{\partial \phi}{\partial \boldsymbol{n}}$ has two values, one for each of the two local orientation of the surface.

Global linear system

The discretization procedure yields the global linear system:

$$\alpha_l u_l = \sum_{j=1}^{N_\Gamma} \{K^d_{lj}\, q_j - K^n_{lj}\, u_j\}$$

By moving nodal unknowns to the left-hand side and keeping known terms on the right-hand side we obtain:

$$\{C_{pl} + K^n_{pl}\} u_p - K^d_{gl}\, q_g = K^d_{pl}\, \overline{q_p} - \{C_{gl} + K^n_{gl}\} \overline{u_g}$$

where $l = 1, ..., N_\Gamma$; $g = 1, ..., N_g$ refers to nodes with a Dirichlet condition on the free surface, and $p = 1, ..., N_p$ refers to nodes with a Neumann condition on the rest of the boundary. **C** is a diagonal matrix containitng the coefficients α_l.

- The linear system obtained is dense
- Iterative methods are typically the choice (GMRES, with a suitable preconditioner)

Time marching: 2^{nd} Ord. Eulerian-Lagrangian formulation

$$\bar{\boldsymbol{R}}(t+\Delta t) = \boldsymbol{R}(t) + \Delta t \frac{\mathrm{D}\boldsymbol{R}}{\mathrm{D}t}(t) + \frac{(\Delta t)^2}{2}\frac{\mathrm{D}^2\boldsymbol{R}}{\mathrm{D}t^2}(t) + \mathscr{O}[(\Delta t)^3]$$

$$\bar{\phi}(\boldsymbol{R}(t+\Delta t)) = \phi(t) + \Delta t \frac{\mathrm{D}\phi}{\mathrm{D}t}(t) + \frac{(\Delta t)^2}{2}\frac{\mathrm{D}^2\phi}{\mathrm{D}t^2}(t) + \mathscr{O}[(\Delta t)^3]$$

- The first order coefficients are given by the b.c. on the free surface, and require the solution of the BIE.
- The second order coefficients are given by Lagrangian time differentiation of the free surface b.c.,

$$\begin{aligned} \frac{D^2\mathbf{R}}{Dt^2} &= \frac{D\mathbf{u}}{Dt} \\ \frac{D^2\phi}{Dt^2} &= \mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} - gw - \frac{1}{\rho}\frac{Dp_a}{Dt} \end{aligned}$$

and require the calculation of $\frac{\partial\phi}{\partial t}$ of $\frac{\partial^2\phi}{\partial t\partial n}$. This is done by solving an analogous BIE for ϕ_t.

The two BIE's share the same geometry, hence they share the same matrix of the linear system, which then needs to be assembled once!

Advanced numerical implementations

- Fast Multipole Method (Fochesato & Dias 2006 [3])
- Adaptive regridding

S. T. Grilli, J. Skourup, and I. A. Svendsen.
An efficient boundary element method for nonlinear water waves.
Engineering Analysis with Boundary Elements, 6:97–107, 1989.

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Proc. Royal Soc. A, 462:2715–2735, 2006.

NUMERICAL METHODS FOR FULLY NONLINEAR FREE SURFACE WAVES

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Short Course on "*Modeling of Nonlinear Ocean Waves*"

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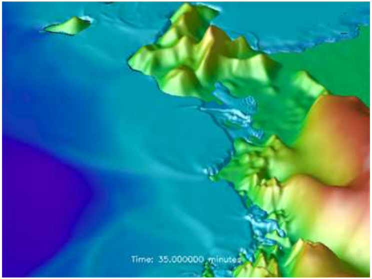

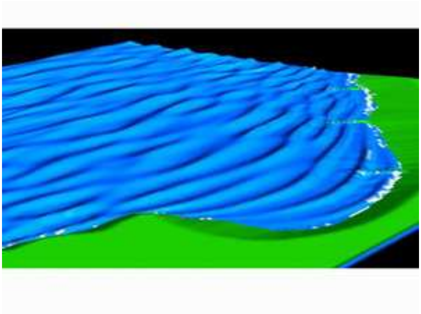

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- Air effect is neglected ($\rho_a \ll \rho_w$)
- Free surface is isobaric

BASIC MODEL:

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Euler equations

with free surface

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$$\nabla \cdot \underline{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

- Momentum conservation:
$$\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} + \frac{\nabla p}{\rho} = \underline{g}$$
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Water wave problem

Potential flow formulation

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- Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$

FIGURE: Laplace

Hamiltonian structure

Petrov (1964) [Pet64]; Zakharov (1968) [Zak68]; Craig & Sulem (1993) [CS93]

Canonical variables:

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$\varphi(\underline{x}, t)$: velocity potential at the free surface

$\varphi(\underline{x}, t) := \phi(\underline{x}, y = \eta(\underline{x}, t), t)$

- Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \varphi}, \qquad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

- Hamiltonian:

$$\mathscr{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla \phi|^2 \, \mathrm{d}y + \frac{1}{2} g \eta^2$$

Application to numerics:

Used in D2N operator methods: [CS93, GN07, XG09]

Luke's variational principles

J.C. Luke, JFM (1967) [Luk67]

- First improvement of the classical Lagrangian $\mathscr{L} := K - \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, \mathrm{d}\underline{x} \, \mathrm{d}t, \qquad \mathscr{L} := \int_{-d}^{\eta} \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy \right) \mathrm{d}y$$

$$\begin{aligned}
\delta\phi: &\quad \Delta\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d, \eta], \\
\delta\phi|_{y=-d}: &\quad \frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d, \\
\delta\phi|_{y=\eta}: &\quad \frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta - \frac{\partial \phi}{\partial y} = 0, \quad y = \eta(\underline{x}, t), \\
\delta\eta: &\quad \frac{\partial \phi}{\partial t} + \tfrac{1}{2} |\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).
\end{aligned}$$

- Water wave problem formulation is recovered from $\mathcal{L}$

Application to numerics:

- Not fully explored...
- Coupled-mode technique by B. & A. (2006) [BA06]

Why this problem is difficult?

Outline of some numerical difficulties

- Problem is highly nonlinear
- Computational domain is unknown ($y = \eta(\underline{x}, t)$ to be determined)
- Formulation is stiff (Hamiltonian structure)
- Taylor expansions involve very high derivatives
- Physical and numerical instabilities
- No dissipation to stabilize computation
- Overturning surface…

To be continued by Claudio. . .

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